Torsional Oscillations of Magnetized Relativistic Stars

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ABSTRACT

Strong magnetic fields in relativistic stars can be a cause of crust fracturing, resulting in the excitation of global torsional oscillations. Such oscillations could become observable in gravitational waves or in high-energy radiation, thus becoming a tool for probing the equation of state of relativistic stars. As the eigenfrequency of torsional oscillation modes is affected by the presence of a strong magnetic field, we study torsional modes in magnetized relativistic stars. We derive the linearized perturbation equations that govern torsional oscillations coupled to the oscillations of a magnetic field, when variations in the metric are neglected (Cowling approximation). The oscillations are described by a single two-dimensional wave equation, which can be solved as a boundary value problem to obtain eigenfrequencies. We find that in the non-magnetized case, typical oscillation periods of the fundamental l=2 torsional modes can be nearly a factor of two larger for relativistic stars than previously computed in the Newtonian limit. For magnetized stars, we show that the influence of the magnetic field is highly dependent on the assumed magnetic field configuration and simple estimates obtained previously in the literature cannot be used for identifying normal modes observationally.

Key words:
Relativity – MHD – stars: neutron – stars: oscillations – stars: magnetic fields – methods: numerical

1 INTRODUCTION

Although the interior composition of relativistic stars is currently still very uncertain, the properties of their solid crust have been studied extensively (Ruderman 1968, see Pethick & Ravenhall 1999 for a recent review). It has been suggested that the presence of a strong magnetic field in a secularly evolving star can cause the crust to fracture, exciting shear waves and leading to the phenomenon of soft gamma repeaters (SGRs) (Cheng et al. 1996, see Thompson 2000 for a recent review of SGRs). In such a scenario, the oscillation modes that appear most favored for excitation are the low-order torsional oscillations of the crust (Duncan 1998). Torsional oscillations differ from spheroidal shear oscillations, in that they are predominantly divergence-free, toroidal velocity oscillations, accompanied by only small density oscillations in the presence of rotation or a magnetic field. Although torsional oscillations of neutron star crusts have a strong potential for observational detection, the study of these modes in the literature has been limited, so far, to a few representative models of the neutron star equilibrium and crust structure, while the magnetic field has only been taken into account in an approximate, Newtonian framework.

Torsional modes may be favored for excitation (compared to spheroidal shear modes) during a starquake, because the restoring force is mainly due to the relatively weak Coulomb forces of the crustal ions (thus requiring much less energy than compressional oscillations) and because the lowest-order torsional oscillation frequency has a relatively long period, which implies a slow damping rate (Duncan 1998). Although no oscillation modes in neutron stars have been detected to date, there is evidence for a 23 ms periodicity in the initial pulse of the 1979 March 5 event (Barat et al. 1983). If confirmed, then torsional modes could become the first normal modes in neutron stars to be detected. As the frequencies of the torsional modes are significantly affected in the presence of a strong magnetic field, accurate frequencies for these modes for various magnetic field strengths and magnetic field configurations are needed, if SGRs are indeed strongly magnetized compact stars.

In the Newtonian framework, torsional modes were first studied by Alterman, Jarosh & Pekeris (1959), applied to oscillations of the Earth's solid crust. That torsional modes could also be excited in neutron star crusts was first proposed by Ruderman (1968) and studied by Hansen & Cioffi (1980), Van Horn (1980), MacDermott et al. 1985 and MacDermott, Van Horn & Hansen (1988). Torsional modes of slowly rotating neutron stars have been studied by Lee & Strohmayer (1996). In general relativity, the theory of torsional oscillations was developed by Schumaker & Thorne (1983), applying the general-relativistic theory of elasticity by Carter & Quintana (1972) (for a discussion, see Priou 1992 and references therein). The only numerical computation of torsional modes for realistic neutron star crusts in general relativity has been presented by Leins (1994).

In the present work, we initiate the study of torsional oscillations in relativistic stars possessing a strong magnetic field. Our theoretical description is based on the study of wave propagation in hydromagnetic media in general relativity by Papadopoulos & Esposito (1982). Previously, the magnetic field effect on the torsional modes has been studied in the Newtonian limit by Carroll et al. (1986) and by Nasiri & Sobuti (1989) (see also Duncan, 1998). Here, we derive the linear perturbation equations governing torsional oscillations in a magnetized star in the relativistic Cowling approximation (McDermott, Van Horn & Scholl, 1983), neglecting the deformation of the equilibrium structure, due to the presence of a magnetic field. For a general axisymmetric magnetic field configuration, the perturbation equations are two-dimensional. Simplified, one-dimensional equations are derived for a special case of the magnetic field configuration. In future work, we plan to include the deformation of the neutron star structure and study more general magnetic field configurations by solving the two-dimensional eigenvalue problem for various magnetic field configurations.

2 THE EQUILIBRIUM CONFIGURATION

The equilibrium structure of a magnetized relativistic star is non-spherical, due to the deformation induced by magnetic field stresses. Self-consistent models of relativistic stars with a strong magnetic field have been constructed by Bocquet et al. (1995) (in the case of rapidly rotating stars) and Gupta et al. (1998) (in the slow-rotation limit). The spherical structure of a nonrotating relativistic star is distorted significantly, only when the magnetic field becomes very large, exceeding $\sim 10^{14}$G (for a magnetic dipole). Although this distortion will certainly modify the eigenfunctions of various oscillation modes, we expect this effect to be smaller than the direct influence of the magnetic field on the eigenfunctions and eigenfrequencies of oscillations, unless the magnetic field becomes exceedingly large. Thus, here we will neglect the distortion induced by the magnetic field and will assume the equilibrium structure to be that of a spherical relativistic star, the space-time of which is described by the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where $\Phi(r)$ and $\Lambda(r)$ are functions of the radial coordinate only. The equilibrium structure is obtained by solving the usual TOV equations (see e.g. Schumaker & Thorne 1983), assuming a perfect-fluid equation of state. Although our purpose is to study oscillations in the solid crust of the star, the bulk properties and local pressure and density profiles in the crust can be obtained accurately, by assuming a perfect fluid. We assume, further, that the crust has an isotropic shear modulus. The shear modulus only appears in the perturbed configuration.

In order to derive the perturbation equations for a magnetized relativistic star, the equilibrium stress-energy tensor is assumed to consist of two parts:

$$T^{ab} = T^{ab(\mathrm{pf})} + T^{ab(\mathrm{M})}, \tag{2}$$

where

$$T^{ab(\mathrm{pf})} = (\epsilon + p)\, u^a u^b + p g^{ab}, \tag{3}$$

and

$$T^{ab(\mathrm{M})} = H^2 u^a u^b + \frac{H^2}{2} g^{ab} - H^a H^b \tag{4}$$

(see Papadopoulos & Esposito, 1982). Above, ϵ and p are the energy density and pressure of the fluid and u^a is the four-velocity of fluid elements

$$u^a = \left[e^{-\Phi(r)}, 0, 0, 0\right], \tag{5}$$

(with $u_a u^a = -1$), while H^a are the components of the magnetic field (with $H^2 = g_{ab} H^a H^b$). The solid crust is assumed to be isotropic and perfectly elastic, so that in equilibrium there is no shear stress contribution in (2). Here, and throughout the paper, we are using the same units as in Papadopoulos & Esposito (1982), i.e. we set $c = G = 1$ and normalize the magnetic field by multiplying H^a by $\sqrt{4\pi}$, unless otherwise noted.

The equations of motion of the fluid are obtained by the projection of the conservation of the stress-energy tensor onto the hypersurface normal to u^a

$$h^c{}_a T^{ab}{}_{;b} = 0, \tag{6}$$

(where $h^c{}_a = g^c{}_a + u^c u_a$ and ";" denotes the covariant derivative compatible with the equilibrium metric g_{ab}), which reads

$$\left(\epsilon + p + H^2\right) u^b u^a{}_{;b} = -\left(p + \frac{H^2}{2}\right)_{;b} h^{ab} + h^a{}_c \left(H^c H^b\right)_{;b}. \tag{7}$$

For perfectly conducting hydromagnetic media, Maxwell's equations $F^{ab}{}_{;b} = 0$ (where F^{ab} is the electromagnetic tensor) take the form

$$\left(u^a H^b - u^b H^a\right)_{;a} = 0, \tag{8}$$

which leads to the following form of the magnetic induction equation

$$H^a{}_{;b} u^b = \left(\sigma^a{}_b + \omega^a{}_b - \frac{2}{3}\delta^a{}_b \Theta\right) H^b + H^b u_{b;c} u^c u^a, \tag{9}$$

where $\Theta = u^b{}_{;b}$ is the expansion of the fluid, $\omega^a{}_b$ is the twist tensor and $\sigma^a{}_b$ is the shear tensor. We assume a zero net electrical surface charge, so that there is no equilibrium electric field. The assumption of infinite conductivity is justified as the magnetic field diffusion timescale is several orders of magnitude larger than the typical timescale for torsional oscillations.

3 THE PERTURBED CONFIGURATION

The various oscillation modes that can be present in a star are studied by linearizing the equations governing the equilibrium configuration and assuming a harmonic time-dependence. Here, we are only interested in the torsional modes of the crust, which do not generate significant variations in the gravitational field of the star. In addition, the low-order quadrupole torsional mode has a long period of the order of 20-40ms. Thus, such an oscillation can be described well by the slow-motion approximation (Shumaker & Thorne 1983). It follows that one can describe the low-order torsional oscillations of the crust by neglecting the variations in the metric, i.e. by setting $\delta g_{ab} = 0$, which is usually called the relativistic Cowling approximation (MacDermott et al. 1983) (this approximation was also used in the description of hydromagnetic waves in general relativity by Papadopoulos & Esposito, 1982). This approximation yields the real part of the eigenfrequencies with a typical accuracy of a few percent, as can be deduced from the results in Leins (1994) and we will employ it throughout this paper. Once the eigenfunction and the real part of the eigenfrequency are computed, estimates for the damping rate of torsional oscillations due to gravitational and electromagnetic wave emission and due to viscous dissipation can be obtained in a perturbative way (see e.g. MacDermott et al. 1988), since the imaginary part of the eigenfrequency is much smaller than the real part.

The linearized version of the equations of motion (7) is

$$\begin{aligned}\left(\epsilon + p + H^2\right) \delta u^a{}_{;b} u^b \quad &= \quad -\left(\delta\epsilon + \delta p + 2H^c \delta H_c\right) u^a{}_{;b} u^b + \left(u^a \delta u_c + \delta u^a u_c\right) \left[H^c H^b - g^{cb}\left(p + \frac{H^2}{2}\right)\right]_{;b} \\ &\quad - \left(\epsilon + p + H^2\right) u^a{}_{;b} \delta u^b + h^a{}_c \left[h^c \delta H^b + \delta H^c H^b - g^{cb}\left(\delta p + H^c \delta H_c\right)\right]_{;b}\end{aligned}$$

$$-h^a{}_c \delta T^{cb\,(\mathrm{S})}{}_{;b}. \tag{10}$$

In (10), the perturbation in the shear stress tensor is

$$\delta T^{(\mathrm{S})}_{ab} = -2\mu \delta S_{ab}, \tag{11}$$

where δS_{ab} is the perturbation in the strain tensor and μ is the isotropic shear modulus. The linearized version of the magnetic induction equation (9) is

$$\begin{aligned}(\delta H^a)_{;c} u^c \quad = \quad & -H^a{}_{;c}\delta u^c + h^{ac} H^d (\delta u_c)_{;d} + h^{ac} u_{c;d} \delta H^d + (u^a \delta u^c + u^c \delta u^a) H^d u_{c;d} - \theta \delta H^a - H^a \delta\theta \\ & -u^a H^c \left[(\delta u_c)_{;b} u^b + u_{c;b}\delta u^b\right] - (H^c \delta u^a + u^a \delta H^c)\, u_{c;b} u^b. \end{aligned} \tag{12}$$

Equations (10) and (12) are the complete set of equations that govern the perturbations of magnetized relativistic stars in the Cowling approximation.

4 THE EIGENVALUE PROBLEM

When the distortion of the equilibrium structure, due to the magnetic field, is ignored, the spherical symmetry of the unperturbed star allows for the oscillations to be decoupled into modes of definite spherical-harmonic indices (l, m) and definite parity. Here, we investigate pure torsional oscillations, which are the normal modes of odd (magnetic-type) parity, $\pi = (-1)^{l+1}$ (Regge & Wheeler, 1957). In spherical symmetry, modes with fixed l but different m yield the same frequency, thus, we will specialize to the case of $m = 0$ only. This means that, we will not study (at this point) the mode-splitting caused by the magnetic field.

When expanded in vector spherical harmonics of definite l and $m = 0$, the odd-parity perturbation in the four-velocity can be written as (see Shumaker & Thorne, 1983)

$$\delta u^\theta \quad = \quad 0 \tag{13}$$

$$\delta u^\phi \quad = \quad e^{-\Phi} \frac{\partial Y(r,t)}{\partial t} b^\phi, \tag{14}$$

where

$$b^\Phi = \frac{1}{\sin\theta} \frac{dP_l(\cos\theta)}{d\theta}, \tag{15}$$

and $Y(r,t)$ is the angular displacement of the oscillating stellar material and $P_l(\cos\theta)$ is the Legendre polynomial of order l. For odd-parity perturbations in a spherical background: $\delta u^t = \delta u^r = \delta\epsilon = 0$.

We assume a harmonic time-dependence of all perturbed variables, as in $Y(r,t) = Y(r)e^{i\omega t}$, where ω is the mode frequency (from now on we drop the time dependence from all perturbed variables). The $\phi-$component of the perturbed equations of motion (10) becomes

$$\begin{aligned} i\omega\left(\epsilon + p + H^2\right) e^{-\Phi}\delta u^\phi \quad = \quad & \delta H^r \left[H^\phi{}_{,r} + H^\phi\left(\Phi_{,r} + \Lambda_{,r} + \frac{3}{r}\right) - H_{r,\phi}\right] + \delta H^\theta \left[H^\phi{}_{,\theta} + \cot\theta H^\phi - H_{\theta,\phi}\right] \\ & + \delta H^\phi \left[H^\phi{}_{,\phi} + 2\cot\theta H^\theta + \frac{2}{r} H^r + \cot\theta H^\phi - H_{\phi,\phi}\right] + H^r \delta H^\phi{}_{,r} \\ & + H^\phi \left[\delta H^r{}_{,r} + \delta H^\theta{}_{,\theta} + \delta H^\phi{}_{,\phi}\right] + H^\theta \delta H^\phi{}_{,\theta} + H^\phi \delta H^\phi{}_{,\phi} - H_r \delta H^r{}_{,\phi} - H_\theta \delta H^\theta{}_{,\phi} \\ & - H_\phi \delta H^\phi{}_{,\phi} - \delta T^{r\phi(\mathrm{S})}_{\ ,r} - \delta T^{\theta\phi(\mathrm{S})}_{\ ,\theta} - \left(\frac{4}{r} + \Phi_{,r} - \Lambda_{,r}\right) \delta T^{r\phi} - 3\cot\theta \delta T^{\theta\phi(\mathrm{S})}, \end{aligned} \tag{16}$$

where

$$\delta T^{(\mathrm{S})}_{r\phi} = -\mu r^2 \sin^2\theta e^{-\Phi} \frac{\partial Y}{\partial r} b^\Phi, \tag{17}$$

and

$$\delta T^{(\mathrm{S})}_{\theta\phi} = -\mu r^2 \sin^2\theta e^{-\Phi} Y \frac{\partial b^\Phi}{\partial\theta}, \tag{18}$$

are the only non-vanishing components of the perturbed shear stress tensor. The oscillating magnetic field components are given algebraically as

$$\delta H^r \quad = \quad 0, \tag{19}$$

$$\delta H^{\theta} = 0, \tag{20}$$

$$\delta H^{\phi} = \frac{e^{\Phi}}{i\omega}\left(H^{r}\delta u^{\phi}{}_{,r} + H^{\theta}\delta u^{\phi}{}_{,\theta}\right). \tag{21}$$

Equation (16) represents a two-dimensional boundary value problem for the eigenfrequency ω and the eigenfunction $Y(r)$. If we substitute (17)-(21) in (16), we arrive at our final expression

$$\begin{aligned}
-\left(\epsilon + p + H^2\right)\omega^2 Y \;=\;& \mu e^{2(\Phi-\Lambda)} Y_{,rr} - (l+2)(l-1)\mu\frac{e^{2\Phi}}{r^2}Y + \left[\left(\frac{4}{r} + \Phi_{,r} - \Lambda_{,r}\right)\mu + \mu_{,r}\right] Y_{,r} e^{2(\Phi-\Lambda)} \\
& + \frac{e^{2\Phi}}{b^{\Phi}}\Bigg\{ b^{\Phi}\Big[2H^{r}\left(H^{\theta}\cot\theta + \frac{1}{r}H^{r}\right)\left(-\Phi_{,r}Y + Y_{,r}\right) + H^{r}H^{r}_{,r}\left(-\Phi_{,r}Y + Y_{,r}\right) \\
& + H^{\theta}H^{r}_{,\theta}\left(-\Phi_{,r}Y + Y_{,r}\right) + (H^{r})^2\left(-\Phi_{,r}Y + Y_{,r}\right)_{,r}\Big] \\
& + b^{\Phi}_{,\theta}\Big[2H^{\theta}Y\left(H^{\theta}\cot\theta + \frac{1}{r}H^{r}\right) + H^{r}\left(H^{\theta}_{,r}Y + H^{\theta}Y_{,r}\right) + H^{\theta}H^{r}\left(-\Phi_{,r}Y + Y_{,r}\right) \\
& + H^{\theta}H^{\theta}_{,\theta}Y\Big] + b^{\Phi}_{,\theta\theta}(H^{\theta})^2 Y\Bigg\},
\end{aligned} \tag{22}$$

In the absence of a magnetic field, separation of variables allows the above equation to be reduced to a one-dimensional equation with respect to the radial coordinate. This is not possible for a general magnetic field configuration. However, as we will show in Section 6, there exist special cases of the magnetic field, for which the problem becomes one-dimensional. As the numerical solution of the two-dimensional problem is not a trivial task, such one-dimensional cases are very useful for obtaining initial order-of-magnitude estimates for the influence of a strong magnetic field on the torsional oscillations.

5 ANALYTIC ESTIMATES

If one assumes a uniform density star with uniform shear modulus μ in the Newtonian limit, one can derive a simple analytic estimate for the influence of the magnetic field on the period of torsional oscillations. In the nonmagnetized case, the period of torsional oscillations is given analytically as

$$P = \frac{2\pi R}{x_n v_s}, \tag{23}$$

where $v_s = \sqrt{\mu/\rho}$ is the speed of sound, ρ is the density of the crust, R is the star's radius and x_n is a constant (see Schumaker and Thorne 1983). In the presence of a magnetic field B, the density ρ is replaced by $\rho + B^2/4\pi$. If one also assumes that the shear modulus μ is augmented by the magnetic field tension $B^2/4\pi$, one easily obtains

$$P = P_0\sqrt{\frac{1+v_A^2}{1+(B/B_\mu)^2}}, \tag{24}$$

where $v_A = B^2/4\pi\rho$ is the Alfvén speed, $B_\mu = (4\pi\mu)^{1/2}$ and P_0 is the oscillation period for a nonmagnetized star. This can be rewritten as

$$P = P_0\sqrt{\frac{1+v_s^2(B/B_\mu)^2}{1+(B/B_\mu)^2}} \simeq P_0[1+(B/B_\mu)^2]^{-1/2}, \tag{25}$$

when $v_s << 1$ (which was also considered by Duncan, 1998). The above estimate implies that the main influence of the magnetic field on the oscillation period is through the magnetic field tension. Based on a power-law fit to the deep-crust equilibrium composition by Negele and Vautherin (1973), Duncan (1998) estimates $B_\mu = 4\times 10^{15}\rho_{14}^{0.4}$G, where $\rho_{14} = \rho/10^{14}$grcm^{-3}. This then yields a significant influence of the magnetic field even at a few times 10^{15}G. As we will show in the next Section, this simple estimate is misleading, as it completely ignores the magnetic field configuration. In practice, the restoring force for the torsional oscillations will not be augmented by the magnetic field tension equally throughout the star, but the influence of the magnetic field on the oscillation will be highly dependent on the correlation between the multipoles of the magnetic field configuration and the quadrupole toroidal velocity field of the torsional mode. For example, as we are showing in the next section, the radial component of the magnetic field has a much weaker influence on the torsional modes than other components. Thus, simple estimates as the one made above, are not sufficiently accurate for obtaining quantitative results that could be used for identifying normal modes in observations.

6 A SPECIAL CASE FOR THE MAGNETIC FIELD

In order to arrive at a one-dimensional boundary value problem, we consider the following magnetic field configuration as a toy-model:

$$H^r = H^r(r) \neq 0, \tag{26}$$

$$H^\theta = 0, \tag{27}$$

For this case, Maxwell's equations reduce to

$$H^r{}_{,r} + \left(\Lambda_{,r} + \frac{2}{r}\right) H^r = 0. \tag{28}$$

The solutions admitted by (28) are of the form

$$H^r = \frac{e^{-\Lambda}}{r^2}\mu_0, \tag{29}$$

where μ_0 is a constant. Although this form of the magnetic field cannot be considered realistic, we will show that this toy-model can be used for obtaining first estimates of the influence of the magnetic field on the oscillation frequencies. In our derivation, we assume that the magnetic field has the above form only inside the solid crust and not throughout the star.

Using (28), one can reduce the two-dimensional boundary value problem (22) to the following one-dimensional form:

$$\begin{aligned} -\omega^2\left(\epsilon + p + H^2\right) r^4 e^{\Lambda-\Phi} Y &= \left\{\left[\mu + \left(e^\Lambda H^r\right)^2\right] e^{\Phi-\Lambda} r^4 Y_{,r}\right\}_{,r} - 2\Phi_{,r} r^4 \left(e^\Lambda H^r\right)^2 Y_{,r} \\ &\quad - \left[(l+2)(l-1)\mu + \left(\Lambda_{,r}\Phi_{,r} - \Phi_{,rr}\right)\left(H^r\right)^2\right] e^{\Phi+\Lambda} r^2 Y. \end{aligned} \tag{30}$$

This equation is a generalization of equation (66a) in Shumaker & Thorne (1983) to the case of the magnetic field configuration assumed above, in the Cowling approximation. We note that both the influence of the equilibrium magnetic field as well as the coupling between torsional and magnetic field oscillations are included in our derivation.

Comparing eqn. (30) to the non-magnetized case, on sees that the magnetic field enters through different terms. First, on the l.h.s. of (30), the total energy density of the hydromagnetic medium becomes $\epsilon + p + H^2$, as one would expect. On the r.h.s., the most important contribution of the magnetic field is the augmentation of the shear modulus μ (in the term involving the second spatial derivative of Y) by the magnetic field term $(e^\Lambda H^r)^2$. This is the only magnetic field term on the r.h.s. that survives in the Newtonian limit, which is in agreement with the Newtonian results in Carroll et al. (1986) (see also Duncan, 1998). The other two contributions of the magnetic field on the r.h.s. are purely relativistic effects. Carroll et al. (1986) only looked at a cylindrical model of the neutron star crust near the magnetic polar cap, with a uniform magnetic field normal to the surface of the crust. The one-dimensional boundary value problem considered in this section is a generalization of Carroll et al.'s cylindrical Newtonian treatment to a spherical geometry and to general relativity.

Eigenvalues of the torsional modes in magnetized stars can be obtained by solving (30) as an eigenvalue problem with zero-traction boundary conditions at the base of the crust and at the surface of the star. For the oscillations of the magnetic field, we assume approximate boundary conditions, neglecting the oscillations in the exterior of the solid crust. For the numerical solution of the above second-order eigenvalue problem, we introduced new variables Y_1 and Y_2 (see Carroll et al. (1986) and Leins (1994) for similar definitions):

$$Y_1 = Y \tag{31}$$

$$Y_2 = \left[\mu + \frac{\left(e^\Lambda H^r\right)^2}{4\pi}\right] \frac{dY_1}{dr} e^{\Phi-\Lambda}, \tag{32}$$

(note that we have restored the factor of 4π dividing the magnetic field). This has the advantages of avoiding the numerical evaluation of the derivative of the shear modulus μ (which is known in a tabular form only, for realistic

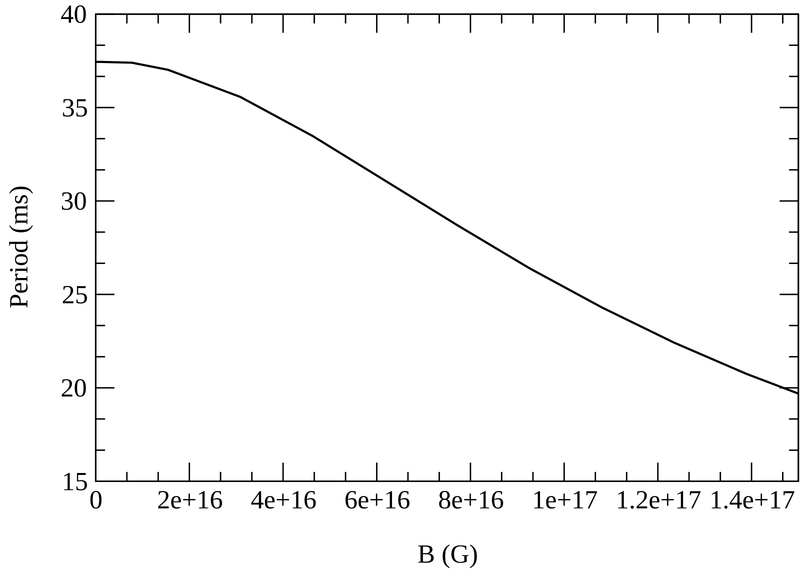

Figure 1. Oscillation period of the fundamental l=2 torsional mode as a function of the magnetic field strength, for a $1.4M_\odot$ relativistic star and for the simple magnetic field configuration considered in Section 6. The influence of the magnetic field is much less than that predicted by simple estimates (see Section 5), which demonstrates that oscillation period is highly dependent on the given magnetic field configuration.

equations of state) and that Y_2 vanishes at the boundaries when zero-traction is assumed. With this definition, the problem of finding the eigenfrequencies ω of torsional modes reduces to solving the system of equations

$$\frac{dY_1}{dr} = \frac{Y_2}{\mu + \frac{\left(e^\Lambda H^r\right)^2}{4\pi}} e^{\Lambda-\Phi}, \tag{33}$$

$$\begin{aligned}\frac{dY_2}{dr} = & -\frac{4}{r} + \frac{4}{r}\frac{(H^r)^2}{4\pi}\frac{e^{2\Lambda}Y_2}{\left[\mu + (e^\Lambda H^r)^2/4\pi\right]} \\ & + \left\{\left[(l+2)(l-1)\mu + \left(\frac{d\Lambda}{dr}\frac{d\Phi}{dr} - \frac{d^2\Phi}{dr^2}\right)\frac{(H^r)^2}{4\pi}\right]\frac{e^{2\Phi}}{r^2} - \omega^2\left(\epsilon + P + \frac{H^2}{4\pi}\right)\right\} e^{\Lambda-\Phi}Y,\end{aligned} \tag{34}$$

with boundary conditions $Y_2 = 0$ at the surface and the base of the crust and Y_1 being any finite number at the base of the crust.

We construct an equilibrium TOV model, using the Wiringa et al. (1988) equation of state (WFF3) matched to the Negele and Vautherin (1973) equation of state at low densities. The star has a gravitational mass of M=1.4 $M_\odot$ and a radius of $R = 10.84$ km. We assume that the crust extends from energy densities of 2.4×10^{14}gr/cm^{-3} down to energy densities of 5×10^{10}gr/cm^{-3}, which corresponds to radii from 9.86km to 10.65km (we truncate the crust at very low densities to avoid numerical difficulties - this does not significantly affect the obtained eigenfrequencies). We note that, apart from the work of Leins (1994), the frequencies of even the non-magnetized case have not been computed for realistic equations of state in general relativity. In all previous Newtonian studies, the eigenfrequency of the fundamental quadrupole torsional mode has been found to be close to 20ms. In our example, this frequency is 37ms. This shows that the structure of a general relativistic star and the relativistic perturbation equations yield torsional frequencies very different from those predicted by the Newtonian equations. For example, part of the difference between the Newtonian and relativistic eigenfrequencies is due to the relativistic term $e^{2\Phi}$, multiplying μ in equation (34). For our model, $e^{2\Phi} = 0.59$ at the base of the crust. This effect increases the fundamental torsional period by a factor of roughly $1/\sqrt{0.59} \simeq 1.3$. The remaining difference between our obtained 37ms fundamental period and the 17.32 ms fundamental period for the most massive model considered in MacDermott et al. (1988) (model NS13T8) can be explained as follows: the model considered in MacDermott et al. (1988) has a mass of 1.326 $M_\odot$, similar to the mass of our present model, but the radius is only 7.853 km, (compared to a radius of 10.65km for our model). The period of the fundamental torsional mode has been shown to be roughly proportional to the star's radius (when the crust is thin) in Hansen and Cioffi (1980), who estimate the $l = 2$ period as $24.5(R/10\text{km})$ms. Using the radius of our model and correcting for general relativity, the above estimate becomes 34.5ms, which is close to our numerically obtained result of 37ms. Thus, a general-relativistic version of Hansen & Cioffi's formula for the period of the fundamental torsional mode is

$$P \simeq 34\text{ms}\left(\frac{R}{10\text{km}}\right), \tag{35}$$

which should hold for $1.4M_\odot$ models constructed with various realistic equations of state, with the possible exception of strange star models with very thin crusts.

Figure 1 shows the period (in ms) of the fundamental quadrupole torsional mode as a function of the magnetic field $B = H^r$. The period is changed by the magnetic field only for values $B > 2 \times 10^{16}$G. In addition, up to several times 10^{16}G, the change in the period does not follow the dependence predicted by the simple estimate (25) (it does so only for larger values of the magnetic field). In contrast the estimate (25) predicted that the magnetic field influences the oscillation period even at a few times 10^{15}G, almost an order of magnitude less than the above numerical result. We conclude that the eigenfrequency of torsional modes in the presence of a strong magnetic field cannot be simply estimated, assuming that the magnetic tension augments the shear modulus in restoring the oscillation, isotropically. Instead, the degree to which the magnetic field modifies the eigenfrequency is highly dependent on the magnetic field configuration and on the correlation of the latter with the toroidal velocity field of the oscillation.

In our example, the magnetic field has only a radial component. A careful inspection of (34) and (22) reveals that the main term that determines the eigenfrequency, $(l+2)(l-1)\mu$, is not augmented by the radial component of the magnetic tension in the Newtonian limit, but only by the azimuthal component. Thus, for a general configuration of the magnetic field, having both radial and azimuthal components, the numerical results of Figure 1 can be considered as a lower limit on the influence of the magnetic field on the torsional modes, while the isotropic simple estimate (25) would be a corresponding upper limit. We expect that the frequencies of torsional modes for realistic magnetic field configurations will have values within these limits.

7 DISCUSSION

We initiate a study of torsional oscillations in relativistic stars possessing a strong magnetic field, based on the treatment of wave-propagation in hydromagnetic media in general relativity by Papadopoulos & Esposito (1982). We derive the linear perturbation equations governing torsional oscillations in a magnetized star in the relativistic Cowling approximation, neglecting the deformation of the equilibrium structure due to the presence of a magnetic field and variations of the metric. The perturbation equations are two-dimensional in the case of a general axisymmetric magnetic field configuration. Simplified, one-dimensional equations are derived for a limiting case of the magnetic field configuration. This allows first estimates for the change in the mode-frequencies, due to the magnetic field, to be obtained. The origin of this change is both due to the equilibrium energy density of the magnetic field and due to the back-reaction of the magnetic field oscillations on the oscillations of the solid crust. Our results are substantially different from a simple isotropic estimate, showing that the influence of the magnetic field on the frequency of torsional oscillations is highly dependent both on the strength and structure of the magnetic field.

The study of torsional oscillations is motivated by the prospect that they could become observable in gravitational waves or in high-energy radiation after a crust fracture is initiated by a strong magnetic field. Work by Duncan (1998) suggests that in this scenario the torsional modes of the crust will be the dominant mode of oscillation, as they do not couple strongly to density variations. In the latter reference it is suggested that an observed periodicity of 21ms in the March 5 event could in fact be the signature of the fundamental quadrupole torsional oscillation of the neutron star's crust. For this to happen, the star must have a very strong magnetic field, since, as we show, this mode has a longer period of 37ms in a typical relativistic nonmagnetized star (contrary to the 20ms estimate derived in previous Newtonian studies).

The correct identification of such periodicities with a specific normal mode requires the computation of mode-frequencies in the presence of a strong magnetic field in full general relativity. A successful identification will significantly constrain the properties of the high-energy equation of state in relativistic stars. As no other mode of oscillation has been identified in neutron stars, to date, torsional modes could be the oscillations to initiate the field of observational neutron-star asteroseismology. Especially a combination of mode-identifications obtained in high-energy radiation observations with mode-identifications in gravitational radiation detections (see Andersson and Kokkotas 1996, Kokkotas, Apostolatos and Andersson 2001) should yield invaluable information on the properties and internal composition of relativistic stars.

In future work, we plan to present detailed numerical results for various realistic neutron star equations of state and for various axisymmetric magnetic field configurations, by numerically solving the full two-dimensional eigenvalue problem.

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